Int. J. Heat Mass Transfer. Vol. 9, pp. 441–451. Pergamon Press 1966. Printed in Great Britain

EQUATIONS FOR THE MASS- OR HEAT-TRANSFER COEFFICIENT IN TURBULENT MOTION

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(*Received* 27 *September* 1965)

Abstract—The renewal models used by several authors for representing the turbulent process in the vicinity of a boundary give only the dependence of the transfer coefficient on the diffusion coefficient or thermal diffusivity. In this paper a number of ways are suggested by means of which this partial information supplied by renewal models may be completed with information of a hydrodynamic nature. The first one makes use of dimensional considerations similar to those used in the classical theory; the second way is based on the instability theory and the third one on certain assumptions concerning the form of the solutions of the equation of motion in the turbulent quasi-steady state. The first way is applied to the mass transfer in the vicinity of a solid boundary and also to the mass transfer in the vicinity of a fluid boundary. The other two ways are applied to the case of mass transfer in a liquid film moving along a vertical wall and to the case of heat transfer between a fluidized bed and a wall.

A comparison between the physical model method and the classical one is also included.

NOMENCLATURE

c, concentration;
c', specific heat of packets of particles;
d, diameter of circle described by ends of stirrer arms;
D, diffusion coefficient;
f, friction factor;
g, acceleration of gravity;
h, heat-transfer coefficient;
I, imaginary part of the coefficient multiplying the time in the expression of the perturbation;
k, mass-transfer coefficient;
k', thermal conductivity in the packets of particles;
m, exponent;
n, number of stirrer revolutions per unit time;
P, consumption power of the agitator;
s, $= u_i/x_0$;
t, time;
u, x-component of fluid velocity;
u_δ, value of u for $Y = \delta$;
u_0, velocity at the beginning of a path of length x_0;
u_i, value of u at the interface;
u_m, average velocity;
v, Y-component of velocity;
v_0, $= (g\delta)^{\frac{1}{2}}$;
v_δ, value of v for $Y = \delta$;
V, volume of liquid in the vessel provided with stirrer;
x, distance along the fluid motion;
x_0, the length of a "fluid particle" path;
y, distance to the fluid interface;
Y, distance to the solid wall.

Greek symbols

β, quantity given by equation (14);
Γ, volumetric liquid flow rate per unit width;
δ, film thickness;
ε, turbulent diffusion coefficient;
ξ, energy dissipated per unit volume in unit time in vicinity of the boundary;
λ_D, dominant wave length;
λ, wave length;
ν, kinematic viscosity of the fluid;
ω, renewal frequency;
ρ, fluid density;
ρ', density of the packets of particles;
σ, surface tension;

τ_0, shear stress at the wall; $\tau_0/\rho = \frac{f}{2} u_m^2$;

φ, friction factor.

1. INTRODUCTION

THE AIM of the present paper is to examine and analyse comparatively the means which may be applied for deducing theoretically equations for the transfer coefficient in the immediate vicinity of a boundary in the case of turbulent motion. These equations may be obtained in two manners. One of them proceeds from the turbulent transport equations obtained by the procedure suggested by Reynolds, while the other, initiated by Danckwerts [1], uses as a starting point a physical model of the turbulent process. Neither method is wholly satisfactory because they either make use of assertions which are arbitrary to a large extent, as for instance the assumptions made concerning the mixing length* and the coefficient of turbulent diffusion, or they are based on a model instead of this latter to result as a consequence of the equations of motion. However, their use is justified in the absence of a more satisfactory theory of turbulence. In the following we shall consider particularly the model method. We shall discuss the other method (the classical method) too, in order to compare the two methods.

In the papers where the second method was applied [1–4], the emphasis was put on establishing a relation between the transfer coefficient and the diffusion coefficient. This leaves the impression that by using the physical model method only this partial information could be obtained, whereas by using the other method full information would be gathered. In the present paper a number of ways are suggested by means of which the partial information supplied by the model may be completed with information of a hydrodynamic nature. The first one makes use of dimensional considerations similar to those applied in the classical theory; the second way is a synthesis of the model and the instability theory; finally, the third way is based on certain assumptions concerning the form of the solutions of the equations of motion in the turbulent quasi-steady state.

The first way has been suggested by the author in his previous papers [5, 6] concerning the mass transfer in the vicinity of a solid boundary. That case will be treated in the present work both for the sake of completeness and, especially, with the aim of comparing the means employed with those of the classical theory. The dimensional considerations way will be applied also to the case of the mass transfer in the vicinity of a fluid boundary and the results will be compared all the time with those of the classical theory. The other two ways will be used for the case of mass transfer in a liquid film moving along a vertical wall and for the case of heat transfer between a fluidized bed and a wall.

2. THE DIMENSIONAL CONSIDERATIONS WAY

2.1 *The case of a solid boundary*

Let us consider a liquid in turbulent motion along a solid boundary (e.g. the turbulent flow of a liquid in a pipe). The movements which take place in the immediate vicinity of the wall have been investigated experimentally by Fage and Townend [7] who have followed up the movements along the wall of dust particles suspended in a liquid with the help of an ultra microscope. They have noticed that groups of dust particles perform large lateral displacements and that the motion which takes place within the short space between two successive lateral displacements is quasi-rectilinear. These experimental facts suggest the representation of the turbulent process in the immediate vicinity of a wall by means of a model proposed by the author [6]; in that model it is considered that laminar boundary layers are formed along the boundary on short portions of length x_0. In other words, the turbulent fluctuations bring liquid elements in contact with the boundary where they move along it over distances of length

* We leave out the fact that the mixing length concept is unsatisfactory for the region in the immediate vicinity of the wall.

x_0 and then dissolve in the bulk of the liquid. The process is repeated at intervals of length x_0. The path-length of the "liquid particles" is undoubtedly subject to fluctuations. As a simplifying assumption it will be considered that such fluctuations are small and consequently that the mass transfer on each path-length takes place under quasi-steady state conditions. The model has been improved [6] in several respects, among others by taking into account the suggestion made by Kramers and Beek [8] and Harriot [4], namely that in the immediate vicinity of the wall there is a thin film of liquid which is not renewed (followed by a layer of liquid wherein renewal takes place). The good agreement with experimental results obtained with the aid of the simple model described above seems to show that the film, though real, is, however, so thin that it does not affect essentially the value of the transfer coefficient. For this reason and since we want only to point out the type of reasoning applied on the model method, we have retained the simple model described above.

It follows from the model that for every interval of length x_0 the equation holding for laminar motion may be applied. Since the path of length x_0 is short, it may be assumed that the thicknesses of the hydrodynamic and diffusion boundary layers which are formed in an element of liquid are smaller than its thickness and therefore that the equation holding for the laminar flow of a semi-infinite fluid along a wall may be used for the mass-transfer coefficient. The following equation is obtained for the mass-transfer coefficient, defined as an average value over the interval of length x_0:

$$k \propto D\left(\frac{u_0}{\nu x_0}\right)^{\frac{1}{2}}\left(\frac{\nu}{D}\right)^{\frac{1}{3}}. \tag{1}$$

Equation (1) gives the value of the mass-transfer coefficient for the region in the immediate vicinity of the wall. Since for liquids we have $\nu/D \gg 1$, the concentration varies appreciably only in the immediate vicinity of the wall and therefore the mass-transfer coefficient between the wall and the liquid is the same as that in the vicinity of the wall.

The classical theory of turbulence suggests that in the immediate vicinity of the wall the motion of the liquid is characterized by the shear stress τ_0, the density ρ and the viscosity ν of the fluid. It follows then that u_0 and x_0 are functions of τ_0, ρ and ν. The number of the physical quantities being four, while that of the dimensions involved three, one can form, for each of the two dependent quantities, a single dimensionless group. Therefore:

$$x_0 \propto \frac{\nu}{(\tau_0/\rho)^{\frac{1}{2}}}, \tag{2}$$

$$u_0 \propto \left(\frac{\tau_0}{\rho}\right)^{\frac{1}{2}}. \tag{3}$$

Equation (1) becomes

$$k \propto \left(\frac{D}{\nu}\right)^{\frac{2}{3}}\left(\frac{\tau_0}{\rho}\right)^{\frac{1}{2}}. \tag{4}$$

There are cases where information concerning the dissipated energy rather than the shear stress τ_0 is available, e.g. mass transfer in a liquid mixed with a stirrer. In such cases it is convenient to replace τ_0 by the quantity ξ_0 (dissipated energy per unit volume in unit time in the vicinity of the solid boundary). Dimensional considerations lead in this case to:

$$x_0 \propto \nu^{\frac{3}{4}}(\xi_0/\rho)^{-\frac{1}{4}}, \tag{5}$$

$$u_0 \propto \left(\frac{\nu\xi_0}{\rho}\right)^{\frac{1}{4}}, \tag{6}$$

and therefore

$$k \propto \left(\frac{D}{\nu}\right)^{\frac{2}{3}}\left(\frac{\nu\xi_0}{\rho}\right)^{\frac{1}{4}}. \tag{7}$$

It should be pointed out that equations (5)–(7) are equivalent to equations (2)–(4) from which they can be deduced if one takes into account that

$$\frac{\xi_0}{\rho} = \frac{1}{\nu}\left(\frac{\tau_0}{\rho}\right)^2.$$

Equations (4) and (7) may be deduced also by applying the classical theory, if it is assumed that the coefficient of turbulent diffusion, ε, in the immediate vicinity of the wall is given by equation:

$$\frac{\varepsilon}{\nu} \propto \left(\frac{Y\sqrt{(\tau_0/\rho)}}{\nu}\right)^3 \equiv \left(\frac{Y(\nu\xi_0/\rho)^{\frac{1}{4}}}{\nu}\right)^3. \quad (8)$$

The fact that ε/ν is a function of

$$\frac{Y\sqrt{(\tau_0/\rho)}}{\nu}\left(\text{or } \frac{Y(\nu\xi_0/\rho)^{\frac{1}{4}}}{\nu}\right)$$

is a consequence of dimensional considerations quite similar to those previously used with respect to x_0 and u_0. The turbulence theory suggests that ε is a function of τ_0, ρ, ν and Y, and dimensional considerations lead to

$$\frac{\varepsilon}{\nu} = F\left(\frac{Y\sqrt{(\tau_0/\rho)}}{\nu}\right) \equiv F\left(\frac{Y(\nu\xi_0/\rho)^{\frac{1}{4}}}{\nu}\right). \quad (8')$$

In support of the value three for the exponent some arguments based on quasi-arbitrary assumptions concerning the mixing length and the velocity fluctuation may be adduced.

Equations (4) and (7) are in excellent agreement with experimental results.* Equation (4), written under the form

$$Nu \propto Re\left(\sqrt{\frac{f}{2}}\right)Pr^{\frac{1}{3}}, \quad (4')$$

has proved to be in agreement with the experimental results of Friend and Metzner [24] regarding heat transfer to a liquid flowing turbulently in a tube and for which $Pr \gg 1$.†

Equation (7), too, is in good agreement with experimental results concerning heat or mass transfer in a fluid mixed by using a stirrer. Assuming that $\xi_0 = P/V$ and taking for the power consumption of the agitator the expression

$$P = \varphi d^5 n^3 \rho, \quad (9)$$

where $\varphi \propto (nd^2/\nu)^{-m}$ with $0{\cdot}20 < m < 0{\cdot}30$, the author established in his paper published in 1954 [10] the equation:

$$\frac{kd^{\frac{1}{4}}V^{\frac{1}{4}}}{D} \propto \left(\frac{\nu}{D}\right)^{\frac{1}{3}}\left(\frac{nd^2}{\nu}\right)^{\frac{3}{4}}\varphi^{\frac{1}{4}}, \quad (10)$$

which is in good agreement with experimental results even as regards the dependence of the transfer coefficient on the speed of revolution of the stirrer. Experimental evidence has shown in fact that $k \propto n^{0{\cdot}7}D^{\frac{2}{3}}$ [8].

The only condition which has to be fulfilled for the model to be used is that the length x_0 of the path should be sufficiently small as compared with the length of the surface along which the liquid flows. (For instance in the case of a sphere or of a set of spheres x_0 should be sufficiently small as compared to the diameter of the particle.)

Equations (4) and (7) have been therefore established: firstly on the basis of a physical model completed with dimensional considerations; secondly on the basis of the equation for the mass flux from the classical theory (where in addition to the coefficient of molecular diffusion we have also the coefficient of turbulent diffusion). In the classical method, dimensional considerations similar to those applied in the previous case are used for establishing an expression for the coefficient of turbulent diffusion and an arbitrary assumption is made regarding the value of a certain exponent. In the model method, the physical model plays the part held by the selection of an exponent in the classical theory. Since the selection of a plausible physical model is less arbitrary than that of a certain value for an exponent, it may be concluded that the physical model method seems logically more

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* Equations (4) and (7) were established initially with the help of the classical theory, the first by Lin *et al.* in 1953 [9] and independently by the author in 1954 [10], and the second by the author, again in 1954, and independently by Calderbank and Moo-Young in 1961 [11]. Calderbank and Moo-Young have established equation (7) on the basis of the theory of isotropic turbulence. The authors have noted quite rightly that the conditions required by isotropic turbulence are very restrictive and that equation (7) should be established on other grounds. The equations have been also established using the model method in references [5] and [6].

† These authors have established equation (4′) by the classical method, using for ε equation (8).

satisfactory than the classical theory. In addition to this, a model supplies a certain understanding of the phenomenon of turbulence, which could not result from the quasi-arbitrary assumptions about the velocity fluctuation, and the rather vague mixing length concept, made in order to justify the selection of the value of the exponent.

2.2 *The case of a fluid boundary*

In the following we shall examine, by the two methods, the mass transfer through the free surface of a liquid film in turbulent motion along a vertical wall, and the mass transfer through the free surface of a liquid mixed by means of a stirrer, applying in both cases procedures similar to those employed in the previous section.

The first problem was examined with the help of the classical theory by Levich [12]. He developed a theory according to which turbulent viscosity and the coefficient of turbulent diffusion become zero on the interface as a consequence of the damping effect of surface tension, and the coefficient of turbulent diffusion increases with the square of the distance to the interface. The assumption that the state of turbulence at the interface is dependent on the characteristic velocity $v_0 = (g\delta)^{\frac{1}{2}}$, surface tension σ and density ρ led him, by using dimensional considerations, to

$$\varepsilon \propto \frac{\rho v_0^3}{\sigma} y^2. \tag{11}$$

It should be noted that for the mass transfer through a fluid boundary, only the relative velocity to the interface contributes to the transfer by convection along a direction normal to the direction of the mean motion of the fluid. But this relative velocity is obviously zero at the interface. For this kinematical reason, the coefficient of turbulent diffusion, which is in fact a consequence of that relative velocity, is nil at the fluid interface. (See also the Appendix.)

The coefficient of turbulent diffusion increases with the velocity v_0 and with distance y to the interface. The surface tension opposing the deformations of the interface attenuates the turbulence, and therefore ε decreases when σ increases. ε depends also on ρ. Since we are dealing with a region remote from a solid boundary, the effect of viscosity seems to be of little importance. Dimensional considerations lead to

$$\frac{\varepsilon \rho v_0}{\sigma} = F_0\left(\frac{y v_0^2 \rho}{\sigma}\right). \tag{11'}$$

The condition that ε should decrease when σ increases requires that the exponent of the group $y v_0^2 \rho/\sigma$ should be higher than unity. If it is assumed that $\varepsilon \propto y^2$, then we get equation (11).

Using equation (11) for ε, we obtain for the transfer coefficient the relation:

$$k \propto \frac{D^{\frac{1}{2}} \rho^{\frac{1}{2}} v_0^{\frac{3}{2}}}{\sigma^{\frac{1}{2}}}. \tag{12}$$

We shall examine the same problem with the help of the physical model described above. It is again considered that owing to turbulence the liquid particles come into contact with the boundary and after travelling a short distance they pass into the bulk of the liquid. However, in contrast with the case of a solid boundary, in the case of a fluid boundary the velocity of the interface is different from zero. For this reason we cannot any longer use equation (1) for the transfer coefficient, but we may use the equation which is valid for a semi-infinite fluid in laminar motion along a plane fluid surface. The last laminar case has been dealt with in literature [13, 14]. Using the equation established by Potter [13], we obtain for the transfer coefficient defined as an average value over the interval of length x_0, the equation

$$k = \frac{2D}{\beta}\left(\frac{74 + 115(u_i/u_0)}{630}\right)^{\frac{1}{2}}\left(\frac{u_0}{\nu x_0}\right)^{\frac{1}{2}}, \tag{13}$$

where β is given by

$$\frac{630\beta^2}{74 + 115(u_i/u_0)}\left\{\frac{3}{10}\frac{u_i}{u_0} + \frac{2}{15}\left(1 - \frac{u_i}{u_0}\right)\beta\right\} = \frac{D}{\nu}. \tag{14}$$

For very small values of u_i/u_0 we get $\beta \approx (\nu/D)^{-\frac{1}{3}}$ and equation (14) leads to equation (1). For values approaching unity, equation (14) leads to

$$k = 1{\cdot}13\left(\frac{Du_i}{x_0}\right)^{\frac{1}{2}}. \qquad (15)$$

This is just the case in the present situation, since in the vicinity of the interface the velocity is practically constant.

The effect of turbulence becomes larger as u_i increases. Consequently x_0 should decrease when u_i increases. On the other hand, surface tension attenuates the turbulence and this causes x_0 to increase with σ. Finally x_0 may be dependent also on the density ρ of the fluid. Dimensional considerations lead to

$$x_0 \propto \frac{\sigma}{\rho u_i^2}. \qquad (16)$$

Equation (15) becomes

$$k \propto \left(\frac{D\rho}{\sigma}\right)^{\frac{1}{2}} u_i^{\frac{3}{2}}. \qquad (17)$$

Equation (17) has the same form as the equation proposed by Levich. It may be written easily in a more useful form for applications if we take into account that, in the case of turbulent motion [15],

$$\delta = 0{\cdot}172\Gamma^{\frac{2}{3}}\, g^{-\frac{1}{3}}. \qquad (18)$$

Equation (18) may be deduced from dimensional considerations if we take into account the fact that viscosity, which acts only in the immediate vicinity of the wall, and surface tension, which acts only in the vicinity of the interface, cannot affect essentially the value of δ.

Putting

$$u_i \propto u_m \equiv \frac{\Gamma}{\delta},$$

equation (17) becomes

$$k \propto \left(\frac{D\rho g}{\sigma}\right)^{\frac{1}{2}} \Gamma^{\frac{1}{2}}. \qquad (19)$$

Equation (19) may also be written

$$\frac{k}{D}\left(\frac{\sigma}{\rho g}\right)^{\frac{1}{2}} \propto \left(\frac{\nu}{D}\right)^{\frac{1}{2}}\left(\frac{\Gamma}{\nu}\right)^{\frac{1}{2}} \equiv Sc^{\frac{1}{2}}\, Re^{\frac{1}{2}}. \qquad (20)$$

As previously stated, the part played by viscosity has not been taken into account since the considered region is sufficiently remote from a solid surface. It should be noticed, however, that the rate of mass transfer may be dependent on the renewal of the interface with fresh elements of liquid having dimensions considerably smaller than the scale of turbulence. The motion of these small elements (which constitute the larger elements having dimensions equal to that of the scale of turbulence) and therefore also their renewal frequency depends on viscosity. If it is considered that viscosity plays a more important part than surface tension,* then dimensional considerations lead to

$$\varepsilon \propto \frac{u_i^2}{\nu}\, y^2, \qquad (21)$$

$$x_0 \propto \frac{\nu}{u_i}, \qquad (22)$$

and for the coefficient of mass transfer we obtain, using either of the two methods, the equation

$$\frac{k\nu^{\frac{2}{3}} g^{-\frac{1}{3}}}{D} \propto \left(\frac{\nu}{D}\right)^{\frac{1}{2}}\left(\frac{\Gamma}{\nu}\right)^{\frac{1}{3}} \equiv Sc^{\frac{1}{2}}\, Re^{\frac{1}{3}}. \qquad (23)$$

It should be noted that, besides dimensional considerations, the classical theory introduces also the assumption that $\varepsilon \propto y^2$. No experimental data are available for making a choice between equations (20) and (23). For this reason we have taken another example which could be treated in the same way as above, but for which experimental results, allowing a choice, are available.

Let us consider a liquid mixed with an agitator and a gas which is absorbed by the liquid through

* The case where both physical constants become effective cannot be considered because the number of variables would be too great for obtaining an equation on dimensional considerations.

the free surface of the latter. The hydrodynamic process taking place in the vicinity of the interface may be characterized by the energy ξ dissipated per unit volume in the vicinity of the interface, the density ρ and either by the surface tension σ or the viscosity ν. In the first case we obtain

$$\varepsilon \propto \left(\frac{\xi}{\rho}\right)^{3/5}\left(\frac{\sigma}{\rho}\right)^{-2/5} y^2, \tag{24}$$

and

$$s \propto \left(\frac{\xi}{\rho}\right)^{3/5}\left(\frac{\sigma}{\rho}\right)^{-2/5}. \tag{25}$$

For the transfer coefficient both methods give

$$k \propto D^{\frac{1}{2}}\left(\frac{\xi}{\rho}\right)^{3/10}\left(\frac{\sigma}{\rho}\right)^{-2/10}. \tag{26}$$

In the second case we get

$$\varepsilon \propto \left(\frac{\xi}{\rho\nu}\right)^{\frac{1}{2}} y^2, \tag{27}$$

and

$$s \propto \left(\frac{\xi}{\rho\nu}\right)^{\frac{1}{2}}. \tag{28}$$

For the transfer coefficient both methods lead to

$$k \propto D^{\frac{1}{2}}\left(\frac{\xi}{\rho\nu}\right)^{\frac{1}{4}}. \tag{29}$$

Putting $\xi = P/V$ and using equation (9) for the power consumption, equations (26) and (29) become

$$k \propto D^{\frac{1}{2}}\, n^{9/10}\, \varphi^{3/10}\, d^{\frac{3}{2}}\, V^{-3/10}\left(\frac{\sigma}{\rho}\right)^{-1/5}. \tag{26'}$$

$$k \propto D^{\frac{1}{2}}\, \nu^{-\frac{1}{4}}\, n^{\frac{3}{4}}\, \varphi^{\frac{1}{4}}\, d^{5/4}\, V^{-\frac{1}{4}}. \tag{29'}$$

The experimental results obtained by Davies *et al.* [16] allow us to make a choice between equations (26) and (29) since they lead to

$$k \propto n^{0.62} \tag{30}$$

and therefore to the conclusion that the part played by viscosity is the more important. However, the conclusion is not certain because equations (26') and (29') have been deduced under the not quite realistic assumption that the dissipated energy is uniformly distributed in the liquid.

As to the comparison of the two methods, all the remarks stated in the preceding section are valid for the case of a fluid boundary too.

An additional argument in favour of the model method is the fact that the method supplies a single equation [equation (13)] from which it follows quite naturally that in the case of a solid boundary $k \propto D^{\frac{2}{3}}$ and in the case of a fluid boundary $k \propto D^{n'}$, the value of n' depending on the ratio u_i/u_0.

3. WAYS BASED ON THE INSTABILITY THEORY AND ON SOLUTIONS OF A CERTAIN TYPE OF THE EQUATION OF MOTION

In the cases examined so far the part played by dimensional considerations is extremely important. There are, however, many instances (particularly for fluid boundaries, as was the case in the preceding examples) where the number of quantities involved is too large for establishing equations on a dimensional basis for the various hydrodynamical parameters. On the other hand, even when dimensional considerations lead to equations there still are constant factors which have to be determined empirically. Other ways have to be found therefore for the solution of the problem.

In this respect two ways seem to us possible for obtaining information regarding the quasi-steady-state turbulence. One of them is based on the theory of instability, while the other starts from certain assumptions regarding the quasi-steady-state turbulence itself.

Within the framework of the first method of approach it may be considered that the quasi-steady-state turbulence represents the final stage of the increase of certain small perturbations applied to a virtual system in laminary motion. Unfortunately the increase of the perturbations can be followed up only when they are sufficiently small, in the validity range of the linearized theory of instability. As soon as the perturbation has become sufficiently large, the non-linear terms of the equation of motion can no longer be neglected, the linear approximation is no longer valid, the calculation becomes very involved and cannot be effected for the time being. However, the final stage of the increase process of the perturbations may be described, at least approximately, with the help of a renewal model. Also, among the small perturbations the one that grows faster (therefore the perturbation for which the real part of the coefficient multiplying the time has the maximum value) imposes itself. It is then reasonable to consider that the characteristics of this perturbation are realized in the final stage. Since, on the one hand the renewal has a quasi-periodic spatial character, while on the other hand consequences of the dominant perturbation are realized in the final stage, we will assimilate the interval x_0 to the wave length of the dominant perturbation.

The second method of approach involves the choice of a certain type of random solution of the equation of motion. In this case too, the difficulties encountered are considerable so that we have to be content in the present with an approximate solution based on the description of the turbulent state with the help of a renewal model. The quasi-periodic character of the renewal process suggests that we look for periodic solutions of the equation of motion and the assimilation of x_0 to their wave length.

As an example we will consider the case of mass transfer in a liquid film in turbulent motion along a vertical wall. Our task is made in this case considerably easier, as both the problem of the stability of the laminar flow of the film [17–19] and the problem of finding periodic solutions of the equation of motion [20] are treated in literature.

Based on the theory of instability, one obtains for the dominant wave length the expression

$$\lambda_D = 8{\cdot}11\left(\frac{\nu\sigma}{\Gamma g\rho}\right)^{\frac{1}{2}} \tag{31}$$

Therefore

$$k = 0{\cdot}397 D^{\frac{1}{2}}\, u_i^{\frac{1}{2}}\left(\frac{\Gamma g\rho}{\nu\sigma}\right) \tag{32}$$

Since

$$u_i \approx 5{\cdot}82\Gamma^{\frac{1}{3}}\, g^{\frac{1}{3}}, \tag{32'}$$

it follows that

$$k \approx 0{\cdot}96 D^{\frac{1}{2}}\, \Gamma^{5/12}\, g^{5/12}\left(\frac{\rho}{\sigma\nu}\right)^{\frac{1}{4}}. \tag{33}$$

Equation (33) may be also written

$$\frac{k}{D}\,\frac{\nu^{\frac{1}{3}}(\sigma/\rho)^{\frac{1}{4}}}{g^{5/12}} \approx 0{\cdot}96\left(\frac{\nu}{D}\right)^{\frac{1}{2}}\left(\frac{\Gamma}{\nu}\right)^{5/12}. \tag{34}$$

It may be noted that equation (34) has an intermediate position between equations (20) and (23) established on the basis of dimensional considerations.

The condition that the equation of motion should have periodic solutions has led Kapitza [20] to the following expression for the wave length of these motions:

$$\lambda = 7{\cdot}5\left(\frac{\nu\sigma}{\Gamma g\rho}\right)^{\frac{1}{2}}. \tag{35}$$

Expression (35) differs from (31) only with respect to the constant factor which this time is somewhat smaller. Consequently the final equation too, differs from equation (34) only by the proportionality constant which is somewhat larger

$$\frac{k}{D}\,\frac{\nu^{\frac{1}{3}}(\sigma/\rho)^{\frac{1}{4}}}{g^{5/12}} \approx \left(\frac{\nu}{D}\right)^{\frac{1}{2}}\left(\frac{\Gamma}{\nu}\right)^{5/12}. \tag{36}$$

It should be noted that in the representation suggested the wave motion and the turbulent motion have the same value for the wave length. The difference consists in the fact that in the case of turbulent motion renewal processes take place at intervals equal to λ, whereas in the case of wave motion the renewal is only partial. (Recently the simpler case of wave motion has been treated by Ruckenstein and Berbente without resorting to a model [21].)

No experimental data are available for checking the equations established above. In any case it may be mentioned that the results obtained closely approach those of the classical theory. However, unlike the classical theory, the methods discussed in the present section take into account to a larger extent the equations of motion of which they are approximate consequences.

The author has been able to check experimentally a single case where the equation for the transfer coefficient was obtained by combining a renewal model with the stability theory, namely the case of the heat-transfer coefficient between a fluidized bed and a wall. In this case the transfer process has been represented by Mickley and Fairbanks [22] with the help of a model where it is considered that packets of particles are exchanged between the bulk of the fluidized bed and the region in the vicinity of the wall. This renewal is determined by the "gas bubbles" passing through the fluidized bed. It is assumed that within the "packets of particles" the void fraction is equal to that corresponding to the minimum fluidization velocity and that the thermal conductivity is equal to that of the corresponding fixed bed. One obtains for the transfer coefficient the equation

$$h = (k'c'\rho'\omega)^{\frac{1}{2}}. \qquad (37)$$

Unlike the models suggested above which have a quasi-steady-state character, the model used by Mickley and Fairbanks has a non-steady-state character since in every "point" at the boundary the packets of particles are replaced with packets from the bulk of the fluidized bed. The quasi-periodic character of this non-steady-state process has induced us to assimilate the renewal frequency ω to $|I|/2\pi$, where I represents the imaginary part of the coefficient multiplying the time in the expression of the perturbation calculated for the dominant wave length.

No details are given here as they have been described in another paper [23]. Mention will be made only that good agreement with experimental data was found, the experimental values of ω being comprised between 4 and 10 s^{-1} while those supplied by our theory are between 5 and 10 s^{-1}.

4. CONCLUSION

It may be said that the method which starts from a model seems more satisfactory from a logical standpoint than the classical theory even when hydrodynamic information is obtained on the basis of dimensional considerations. Moreover, in contrast to the classical theory, this approach allows us to take into account the equations of motion to a larger extent; the results obtained with the help of the other two ways described above can be considered as approximate consequences of the equation of motion.

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APPENDIX

In the present paper it has been assumed that $\varepsilon = 0$ at the fluid boundary. The turbulent kinematic viscosity, however, has on the interface values different from zero, since the velocity component along a direction perpendicular to the direction of mean motion is not nil for a moving boundary. It should be noted, however, that for the mass transfer through a moving boundary only the velocity relative to the interface contributes to the transfer by convection in the direction normal to the wall. This relative velocity is zero on the interface and consequently the coefficient of turbulent diffusion vanishes at the interface. This conclusion may be also arrived at in the following quantitative way. The equation of convective diffusion for a film of liquid in turbulent motion along a wall has the form

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial Y} = D\frac{\partial^2 c}{\partial Y^2}, \qquad \text{(A1)}$$

where Y represents the distance from the wall.

For mass transfer through a moving interface we are concerned in following up a process that takes place with respect to another reference system, namely a reference system attached to the interface

$$y = \delta - Y, \quad x = x, \quad t = t. \qquad \text{(A2)}$$

But in the case of turbulent motion the thickness δ of the film is (a random) function of x and t. For this reason the change of variable mentioned above transforms equation (A1) in:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + \left(\frac{\partial \delta}{\partial t} + u\frac{\partial \delta}{\partial x} - v\right)\frac{\partial c}{\partial y} = D\frac{\partial^2 c}{\partial y^2} \qquad \text{(A3)}$$

But the kinematic condition at the interface leads to

$$\frac{\partial \delta}{\partial t} + u_\delta\frac{\partial \delta}{\partial x} - v_\delta = 0. \qquad \text{(A4)}$$

Equation (A4) points to the conclusion that the term

$$\left(\frac{\partial \delta}{\partial t} + u\frac{\partial \delta}{\partial x} - v\right)\frac{\partial c}{\partial y}$$

of equation (A3), which represents the effect of convection along a direction perpendicular to the wall, is nil at the interface. It is therefore natural that turbulent diffusion, which derives in fact from this term, should be nil at the interface.

Résumé—Les modèles de renouvellement utilisés par plusieurs auteurs pour réprésenter le processus turbulent au voisinage d'une paroi donne seulement la dépendance du coefficient de transport en fonction du coefficient de diffusion ou de la diffusivité thermique. Dans cet article, on suggère un certain nombre de méthodes au moyen desquelles on peut compléter à l'aide de l'hydrodynamique les renseignements partiels donnés par les modèles de renouvellement. La première méthode utilise des considérations dimensionnelles semblables à celles de la théorie classique; la deuxième est basée sur la théorie de l'instabilité et la troisième sur certaines hypothèses concernant la forme des solutions de l'équation du mouvement dans le régime turbulent quasi-permanent. La première méthode est appliquée au transport de masse au voisinage d'une paroi solide et également au transport de masse au voisinage d'une surface fluide. Les deux autres méthodes sont appliquées au cas du transport de masse dans un film liquide s'écoulant le long d'une paroi verticale et au cas du transport de chaleur entre un lit fluidisé et une paroi.

Une comparaison entre la méthode du modèle physique et la méthode classique est également donnée.

Zusammenfassung—Die Erneuerungsmodelle die von verschiedenen Autoren zur Darstellung des turbulenten Prozesses in der Umgebung einer Begrenzung verwendet werden geben nur die Abhängigkeit des Übergangskoeffizienten vom Diffusionskoeffizienten oder der thermischen Diffusivität. In vorliegender Arbeit werden eine Reihe von Möglichkeiten vorgeschlagen wodurch diese von den Erneuerungsmodellen erhaltene Information mit Angaben über die hydrodynamische Natur vervollständigt werden kann. Die erste Möglichkeit verwendet Dimensionsbetrachtungen ähnlich der klassischen Theorie die zweite beruht auf der Instabilitätstheorie und die dritte auf gewissen Annahmen für die Form der Lösungen der Bewegungsgleichungen im turbulenten quasistationären Zustand. Die erste Möglickheit wird auf den Stoffübergang in der Nähe sowohl einer festen Begrenzung als auch einer flüssigen angewandt. Die beiden anderen Möglichkeiten werden für den Fall des Stoffübergangs in einem Flüssigkeitsfilm der sich entlang einer senkrechten Wand bewegt und für den Wärmeübergang zwischen einem Fliessbett und der Wand verwendet. Ein Vergleich zwischen der physikalischen Modellmethode und der klassischen ist ebenfalls gemacht.

Аннотация—Модели восстановления, которыми некоторые авторы пользуются для описания турбулентного процесса вблизи границы раздела, позволяют определить только влияние коэффициента диффузии массы или тепла на коэффициент теплообмена. В данной статье предложен ряд методов, с помощью которых эти неполные сведения можно дополнить данными гидродинамического характера. В основе первого метода лежат принципы размерности, обычно используемые в классической теории, второго—теория неустойчивости, третьего-определенные закономерности, вытекающие из допущения относительно вида решений уравнения движения для турбулентного квазистационарного состояния. Первый метод применяется в случае переноса массы вблизи твердой границы, а также переноса массы вблизи границы жидкости. Два других метода применяются в случае переноса массы в движущейся пленке жидкости на вертикальной стенке и переноса массы между псевдоожиженным слоем и стенкой.

Проведено сравнение метода физической модели с классическим методом.